

# ON DEGENERATE SECTIONS OF VECTOR BUNDLES

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**ABSTRACT.** We consider the locus of sections of a vector bundle on a projective scheme that vanish in higher dimension than expected. We show that after applying a high enough twist, any maximal component of this locus consists entirely of sections vanishing along a subscheme of minimal degree. In fact, we will give a more refined description of this locus, which will allow us to deduce its limit in the Grothendieck ring of varieties.

## 1. INTRODUCTION

Given a vector bundle  $V$  on a connected projective scheme  $X$ , we would expect a general section  $s \in H^0(V)$  to vanish on a locus of codimension  $\text{rank}(V)$  in  $X$ . If we regard  $H^0(V)$  as an affine space, then there is a closed locus  $D(V) \subset H^0(V)$  corresponding to sections that vanish in higher dimension. We are interested in basic questions about this locus, for example:

**Question 1.1.** *What is the dimension of  $D(V)$ ? What are the components, and what can we say about them?*

The purpose of this paper is to give a clean answer after a twist by a high tensor power of  $\mathcal{O}_X(1)$ . For example, we can show

**Theorem 1.2.** *There exists  $N_0$  such that for all  $N \geq N_0$ , every section  $s \in D(V(N))$  contained in a component of maximal dimension must vanish on a codimension  $\text{rank}(V) - 1$  variety of minimal degree.*

In the special case where  $X = \mathbb{P}^r$  and  $V = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_k)$  is totally split, then Theorem 1.2 specializes to

**Corollary 1.3.** *Given degrees  $d_1, \dots, d_k$ , there exists  $N_0$  such that for  $N \geq N_0$ , the unique largest component of the locus*

$$D(\mathcal{O}_{\mathbb{P}^r}(d_1 + N) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^r}(d_k + N)) \subset H^0(d_1 + N) \oplus \cdots \oplus H^0(d_k + N)$$

*of  $k$ -tuples of hypersurfaces that fail to intersect properly is the locus where the hypersurfaces all contain a codimension  $k - 1$  linear space.*

Previously, the author has obtained a quantitative version of Corollary 1.3, where the results are cleaner when  $k = r$  [14]. Even in this special case of a total split vector bundle on projective space, there are easy counterexamples to the conclusion of Theorem 1.2 if we don't apply a large twist. For example, in the case  $X = \mathbb{P}^4$  and  $V = \mathcal{O}(2) \oplus \mathcal{O}(2)$ , it is fewer conditions for the two quadrics to be equal than for them to contain a common hyperplane.

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**1.1. Informal summary of results.** Theorem 1.2 will follow from a more refined description. We will show that in the limit as  $N$  grows large, the components of  $D(V(N)) \subset H^0(V(N))$  are indexed by the components of the restricted Hilbert scheme of codimension  $\dim(V) - 1$  subschemes. The larger components of  $D(V(N))$  correspond to components parameterizing subschemes over which  $V$  has a smaller Hilbert polynomial.

As a corollary of our analysis, the class  $[D(V(N))]$  in the Grothendieck ring of varieties (divided by an appropriate power of  $[\mathbb{A}^1]$ ) converges as  $N \rightarrow \infty$ .

**1.2. Summary of results.** We now formally state our results. Given two polynomials  $p(t), q(t) \in \mathbb{Q}[t]$ , we say  $p$  *dominates*  $q$  if and only if  $\lim_{t \rightarrow \infty} p(t) - q(t) = \infty$ . We say  $p(t)$  and  $q(t)$  are *equivalent* if neither  $p(t)$  or  $q(t)$  dominates the other. We denote this by  $p(t) \sim q(t)$ . Put another way,  $p(t) \sim q(t)$  if they differ only in their constant terms.

Let  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}$  denote the open locus of the Hilbert scheme of  $X$  parameterizing codimension  $\text{rank}(V) - 1$  geometrically integral subschemes. Let

$$\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}(V, p(t)) \subset \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}$$

denote the connected components of  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}$  parameterizing subschemes  $Z$  whose Hilbert polynomial  $\chi(V|_Z(t))$  with respect to  $Z$  is *equal* to  $p(t)$ .

Let  $S$  be the set of polynomials in  $\mathbb{Q}[t]$  containing all  $p$  for which  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}(V, p)$  is nonempty. By Chow's finiteness theorem, there are only finitely many components of  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}$  parameterizing varieties of some fixed degree. Therefore, we can pick a sequence  $p_1, p_2, \dots$  of polynomials in  $S$  such that every  $p \in S$  is equivalent to  $p_i$  for some  $i$  and  $p_i$  is dominated by  $p_{i+1}$  for each  $i$ .

Given a polynomial  $p(t)$  of degree  $\dim(X) - \text{rank}(V) + 1$ , we can consider the incidence correspondence

$$\begin{array}{ccc} & \tilde{D}(V, p(t)) & \\ \swarrow \pi_1 & & \searrow \pi_2 \\ D(V) & & \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}(V, p(t)) \end{array}$$

where  $\tilde{D}(V, p(t))$  parameterizes pairs  $(s, [Z]) \in H^0(V) \times \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}(V, p(t))$ , where  $s$  vanishes on  $Z$ . The scheme  $\tilde{D}(V, p(t))$  has a canonical scheme theoretic structure as an open subset of a relative Hilbert scheme, but we are only interested in it as an algebraic set. Let  $D(V, p(t))$  be the constructible set that is the image of  $\pi_1$ .

Note that for fixed  $p(t)$ , there exists  $N_0$  such that for all  $N \geq N_0$ ,  $\overline{D(V, p(t))}$  is precisely the sections vanishing on some element of the closure of  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}(V, p(t))$  by Corollary 4.3.

**Theorem 1.4.** *There exists  $N_0$  such that for all  $N \geq N_0$ , a component of  $D(V(N))$  of maximal dimension is in  $\overline{D(V(N), p(t))}$  for  $p(t) \sim p_1(t)$ .*

More generally, for each  $m \geq 0$ , there exists  $N_0$  such that for  $N \geq N_0$ , a component of

$$D(V(N)) \setminus \bigcup_{i=1}^m \bigcup_{p(t) \sim p_i(t)} D(V(N), p(t))$$

of maximal dimension is in  $\overline{D(V(N), p(t))}$  for  $p(t) \sim p_{m+1}(t)$ .

**1.2.1. Convergence in the Grothendieck ring.** Since it turns out that  $D(V, p(t))$  is isomorphic as an algebraic set to a vector bundle over  $\widehat{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}(V, p(t))$  away from a set whose codimension grows with  $N$ , it is easy to conclude convergence in the Grothendieck ring from Theorem 1.4.

Let  $\mathcal{M}$  be the Grothendieck ring of varieties. Given a finite type  $K$ -scheme  $Z$ , its class in the Grothendieck ring is denoted by  $[Z]$ . Let  $\mathbf{L} = [\mathbb{A}^1]$ . Then, we have a filtration

$$\cdots \subset F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset \mathcal{M}_{\mathbf{L}}$$

by dimension, and we can complete to obtain a ring  $\widehat{\mathcal{M}}_{\mathbf{L}}$  [2, Section 1.5]. For more details on the Grothendieck ring and another example of a limit in  $\widehat{\mathcal{M}}_{\mathbf{L}}$ , see [15].

**Corollary 1.5.** *If  $p_V(t)$  is the Hilbert polynomial  $\chi(X, V(t))$ , we have*

$$\lim_{N \rightarrow \infty} \frac{[D(V(N))]}{\mathbf{L}^{p_V(N) - p_1(N)}} = \sum_{p(t) \sim p_1(t)} [\widehat{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}(V, p(t))] \mathbf{L}^{p_1(N) - p(N)} \quad (\text{in } \widehat{\mathcal{M}}_{\mathbf{L}})$$

and more generally for  $m \geq 0$ , we have

$$\lim_{N \rightarrow \infty} \frac{[D(V(N)) \setminus \bigcup_{i=1}^m \bigcup_{p(t) \sim p_i(t)} D(V(N), p(t))]}{\mathbf{L}^{p_V(N) - p_{m+1}(N)}} = \sum_{p(t) \sim p_{m+1}(t)} \frac{[\widehat{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}(V, p(t))]}{\mathbf{L}^{p(N) - p_{m+1}(N)}}.$$

Note that by definition,  $p_{m+1}(N) - p(N)$  is constant for  $p(t) \sim p_{m+1}(t)$ , so the right side of the limits in the statement of Corollary 1.5 is constant in  $N$ .

Finally, we note that it is no harder to generalize Theorem 1.4 and Corollary 1.5 to the locus of sections that vanish in dimension  $a$  more than expected for  $a$  a positive integer (so Theorem 1.4 and Corollary 1.5 are stated in the case  $a = 1$ ), and this is the generality in which we will work for the rest of the paper. The appropriate generalizations of Theorem 1.4 and Corollary 1.5 are stated in Theorem 7.1 and Theorem 7.2.

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## 3. DEFINITIONS

Throughout the main body of the paper, we will fix a projective scheme  $X$  and a vector bundle  $V$  on  $X$ . We will work over an algebraically closed field  $K$  of arbitrary characteristic. Without loss of generality, we can and will replace  $V$  by a twist so that it is 0-regular in the sense of Castelnuovo-Mumford regularity. In particular,  $V$  is globally generated. Unless otherwise specified,  $a$  is a positive integer.

Following Section 1.2, we define the following

**Definition 3.1.** Let  $\text{Hilb}_X$  be the Hilbert scheme of subschemes of  $X$  and  $\widetilde{\text{Hilb}}_X \subset \text{Hilb}_X$  denote the locus parameterizing geometrically irreducible subschemes, which is open by [4, IV 12.2.1(x)]. To indicate dimension, we let  $\text{Hilb}_X^c$  and  $\widetilde{\text{Hilb}}_X^c$  denote the restriction to the connected components parameterizing subschemes of dimension  $c$ .

**Definition 3.2.** Given a polynomial  $p(t) \in \mathbb{Q}[t]$ , let  $\widetilde{\text{Hilb}}_X^c(V, p(t)) \subset \widetilde{\text{Hilb}}_X^c$  denote the connected components of  $\widetilde{\text{Hilb}}_X^c$  parameterizing subschemes  $Z$  where  $\chi(V|_Z(t)) = p(t)$ .

**Definition 3.3.** Let  $p_V \in \mathbb{Q}[t]$  denote the Hilbert polynomial  $\chi(X, V(t))$ .

**Definition 3.4.** Let  $D(V, a) \subset H^0(V)$  denote the closed locus of sections  $s \in H^0(V)$  such that  $\{s = 0\}$  is codimension at most  $\text{rank}(V) - a$  in  $X$ .

**Definition 3.5.** Let  $\tilde{D}(V, a, p(t))$  consist of pairs  $(s, [Z]) \in H^0(V) \times \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p(t))$ , where  $s$  vanishes on  $Z$ . The locus  $\tilde{D}(V, p(t)) \subset H^0(V) \times \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p(t))$  is closed and it can even be given a canonical scheme structure as an open subset of a relative Hilbert scheme [1, Lemma 7.1].

As before, we have the incidence correspondence

$$\begin{array}{ccc} & \tilde{D}(V, a, p(t)) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ D(V, a) & & \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t)) \end{array}$$

**Definition 3.6.** Given  $p(t) \in \mathbb{Q}[t]$ , let  $D(V, a, p(t))$  be the constructible subset  $\pi_1(\tilde{D}(V, a, p(t)))$  of  $H^0(V)$ .

**Definition 3.7.** Let  $\overline{\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p(t))}$  be the closure of  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p(t))$  in  $\text{Hilb}_X$ .

**Definition 3.8.** Like in Definition 3.5, let  $\tilde{D}(V, a, p(t))^{\text{cl}}$  denote the pairs  $(s, [Z]) \in H^0(V) \times \overline{\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p(t))}$  where  $s$  vanishes on  $Z$ .

$$\begin{array}{ccc} & \tilde{D}(V, a, p(t))^{\text{cl}} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ D(V, a) & & \overline{\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t))} \end{array}$$

**Definition 3.9.** Like in Definition 3.6, let  $D(V, a, p(t))^{\text{cl}}$  be the closed subset  $\pi_1(\tilde{D}(V, a, p(t))^{\text{cl}})$  of  $H^0(V)$ .

**Definition 3.10.** Given  $Z \subset X$  and a vector bundle  $V'$  on  $X$ , define the Hilbert function  $h_{Z, V'}$  to be

$$h_{Z, V'}(n) := \dim(\text{im}(H^0(V'(n)) \rightarrow H^0(V'(n)|_Z))).$$

In particular, Definition 3.10 depends on our ambient projective scheme  $X$ .

**3.1. Constructible sets.** Since  $D(V, a, p(t))$  in Definition 3.6 is a constructible set, we will need to work with constructible sets. To take the dimension of a constructible set, it suffices to either look at the generic points or take the closure.

**Definition 3.11.** If  $A \subset X$  is a constructible set, then  $\dim(A) := \dim(\overline{A})$ .

**Lemma 3.1.** *If  $f : X \rightarrow Y$  is a morphism of finite type  $K$ -schemes,  $A \subset X$  and  $B \subset Y$  constructible sets, and  $\dim(f^{-1}(b) \cap A) < c$  for all  $c \in B$ , then*

$$\dim(A) \leq \dim(B) + c.$$

*If  $\dim(f^{-1}(b)) = c$  for all  $c \in B$ , then equality holds.*

*Proof.* Apply the usual theorem on fiber dimension at the generic points of the components of  $\overline{A}$  to  $f|_{\overline{A}} : \overline{A} \rightarrow \overline{B}$ .  $\square$

**Definition 3.12.** If  $A \subset B$  are constructible subsets of a scheme  $Y$ , then the codimension of  $A$  in  $B$  is defined to be  $\dim(B) - \dim(A)$ . If  $A$  is empty, then the codimension is  $\infty$ .

#### 4. NAIVE EXPECTATION

We will describe a naive argument that show why we might expect Theorem 1.4 and Corollary 1.5 to be true. We then will identify the main obstacles that need to be overcome to turn this naive argument into a proof.

**Proposition 4.1.** *Given  $p(t) \in \mathbb{Q}[t]$  of degree  $\dim(X) - \text{rank}(V) + a$ , there exists  $N_0$  dependent on  $p(t)$  such that for all  $N \geq N_0$ ,*

$$\widetilde{D}(V(N), a, p(t))^{cl} \rightarrow \overline{\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p(t))}$$

*is set-theoretically a vector bundle of rank  $p_V(N) - p(N)$ .*

*Proof.* First, we can choose  $N_0$  large enough so that for each  $[Z] \in \overline{\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}(V, p(t))}$ , the Hilbert function and Hilbert polynomial agree for  $N \geq N_0$ ,  $h_{Z,V}(N) = p(N)$  and all the higher cohomologies of  $V(N)|_Z$  vanish.

To see this, for  $[Z] \in \overline{\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + 1}(V, p(t))}$  such that  $V(N)|_Z$  or  $V(N) \otimes I_Z$  has higher cohomology, we can increase  $N_0$  to kill the higher cohomology, and then apply Noetherian induction. Alternatively, this also follows from [3, Proposition 4.1] applied to both  $V|_Z$  and  $V \otimes I_Z$ , where  $I_Z$  is the ideal sheaf. There is a globally generated hypothesis required to apply the Proposition, but  $V$  is globally generated by assumption and we can apply the usual regularity theorem [3, Theorem 2.7] to  $I_Z$  and the fact 0-regular implies globally generated [11, Theorem 1.8.3(i)].

Let

$$\pi : \mathcal{Y} \rightarrow \left( \overline{\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p(t))} \right)^{\text{red}}$$

be the universal family restricted to the reduction of the Hilbert scheme and  $\rho : \mathcal{Y} \rightarrow X$  the canonical map that is an embedding on each fiber. By Grauert's theorem,  $\pi_* \rho^* V(N)$  is a vector bundle of rank  $p(N)$ . Then, we can pull back  $H^0(V(N)) \otimes \mathcal{O}_X \rightarrow V$  to get  $H^0(V(N)) \otimes \mathcal{O}_{\mathcal{Y}} \rightarrow \rho^* V$ . Pushing forward by  $\pi_*$  gives us

$$H^0(V(N)) \otimes \mathcal{O}_{\left(\widetilde{\text{Hilb}}_X^{\dim(X)-\text{rank}(V)+a}(V, p(t))\right)^{\text{red}}} \longrightarrow H^0(V(N)) \otimes \pi_* \mathcal{O}_Y \longrightarrow \pi_* \rho^* V(N)$$

$\phi$

Restricted to a point  $[Z] \in \left(\widetilde{\text{Hilb}}_X^{\dim(X)-\text{rank}(V)+a}(V, p(t))\right)^{\text{red}}$ ,  $\phi$  is the restriction map  $H^0(V(N)) \rightarrow H^0(V(N)|_Z)$ . From our choice of  $N_0$ , this restriction map must be surjective. Therefore,  $\phi$  is a surjective map of vector bundles and  $\ker(\phi)$  is a vector bundle.

Over a point  $[Z] \in \left(\widetilde{\text{Hilb}}_X^{\dim(X)-\text{rank}(V)+a}(V, p(t))\right)^{\text{red}}$ ,  $\ker(\phi)$  is precisely the sections of  $H^0(V(N))$  that vanish on  $Z$ . Therefore, if we regard  $|\ker(\phi)|$  as the affine bundle corresponding to  $\ker(\phi)$ , we see  $|\ker(\phi)|$  and  $\tilde{D}(V(N), a, p(t))^{\text{cl}}$  agree set-theoretically.  $\square$

From Proposition 4.1, we see

**Corollary 4.2.** *Given  $p(t)$  with  $\widetilde{\text{Hilb}}_X^{\dim(X)-\text{rank}(V)+a}(V, a, p(t))$  nonempty,  $\tilde{D}(V(N), a, p(t))^{\text{cl}}$  is dimension*

$$p_V(N) - p(N) + \dim(\widetilde{\text{Hilb}}_X^{\dim(X)-\text{rank}(V)+a}(V, a, p(t)))$$

for all  $N \geq N_0$  with  $N_0$  depending on  $p(t)$ .

**Corollary 4.3.** *Given  $p(t)$ , there is  $N_0$  dependent on  $p(t)$  such that for all  $N \geq N_0$ ,  $\overline{D(V(N), a, p(t))} = D(V(N), a, p(t))^{\text{cl}}$ .*

From Corollary 4.2, we see that if  $p(t)$  is dominated by  $q(t)$ , then there exists  $N_0$  such that for all  $N_0 \geq N$ ,

$$\dim(\tilde{D}(V(N), a, p(t))^{\text{cl}}) > \dim(\tilde{D}(V(N), a, q(t))^{\text{cl}})$$

and the difference grows asymptotically with  $N$ .

Therefore, to conclude Theorem 1.4 and Corollary 1.5, we still need to accomplish two things. First, we need to show that the map  $\tilde{D}(V(N), a, p(t)) \rightarrow D(V(N), a, p(t))$  is an isomorphism over a set whose codimension grows with  $N$  and the codimension of  $D(V(N), a, p(t)) \cap D(V(N), a, q(t))$  inside of  $D(V(N), a, p(t))$  grows with  $N$ . This we will solve with another incidence correspondence in Section 5.

The more serious problem is that we have an issue with the order of quantifiers. There are countably many polynomials  $p(t)$  we need to consider and the constant  $N_0$  in Proposition 4.1 depends on  $p(t)$ . To illustrate this problem, Proposition 4.1 still holds in the case  $a = 0$ , but  $D(V(N), 0)$  is all of  $H^0(V(N))$ , the point being that for each  $N$ ,  $D(V(N), 0) = D(V(N), p(t))^{\text{cl}}$  for some  $p(t)$  that depends on  $N$ . For example, if  $X = \mathbb{P}^3$  and  $V = \mathcal{O}(d_1) \oplus \mathcal{O}(d_2)$ , then a section of  $V(N)$  will vanish along a complete intersection of type  $(d_1 + N, d_2 + N)$ , but in general it will not vanish along a linear space.

We have to show that when  $a > 0$  such a thing cannot happen, and we will do this by proving a bound that works for  $N \geq N_0$ , where  $N_0$  does not depend on  $p(t)$ , and the dimension of  $\widetilde{\text{Hilb}}_X^{\dim(X)-\text{rank}(V)+a}(V, a, p(t))$  does not appear as a term in the bound in Section 6.

## 5. CASE OF LOW DEGREE

To deal with the first issue outlined at the end of the previous section, we show

**Proposition 5.1.** *Given  $p(t)$  of degree  $\dim(X) - \text{rank}(V) + a$ , there exists  $N_0$  depending on  $p(t)$  such that for all  $N \geq N_0$ , there is a closed subset  $E(N) \subset \tilde{D}(V(N), a, p(t))$ , whose codimension in  $\tilde{D}(V(N), a, p(t))$  is bounded from below by  $P(N)$ , for  $P(t)$  a polynomial with the same degree and leading coefficient as  $p(t)$ , such that the fibers of*

$$\pi : \tilde{D}(V(N), a, p(t)) \setminus E(N) \rightarrow D(V(N), a, p(t)) \setminus \pi(E(N))$$

*are a single reduced point.*

*Proof.* Since the fibers of  $\pi$  have dimension at most  $\dim(\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t)))$ , it suffices to find a closed subset  $F(N) \subset D(V(N), a, p(t))$  of high codimension and let  $E(N) = \pi^{-1}(F(N))$ . To do this, let  $\mathcal{H}^{[2]}$  be the Hilbert scheme of length two subschemes of  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t))$ . If  $\mathcal{Z} \rightarrow \mathcal{H}^{[2]}$  is the universal family, there is an induced map  $\rho : \mathcal{Z} \rightarrow \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t))$ . Let  $\mathcal{Y} \rightarrow \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t))$  be the universal family and consider the composition  $\rho^* \mathcal{Y} \rightarrow \mathcal{Z} \rightarrow \mathcal{H}^{[2]}$ .

$$\begin{array}{ccc} X & \xleftarrow{\tilde{\rho}} & \rho^* \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ & & \downarrow & & \downarrow \\ & & \phi \left( \mathcal{Z} \right) & \xrightarrow{\rho} & \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t)) \\ & & \downarrow & & \\ & & \mathcal{H}^{[2]} & & \end{array}$$

Now, let  $\tilde{D}_2(V(N), a, p(t)) \subset H^0(V) \times \mathcal{H}^{[2]}$  consist of pairs  $(s, [Z])$  such that  $s$  pulled back to  $(\tilde{\rho}^* V)|_{\phi^{-1}([Z])}$  vanishes. Equivalently,  $s$  vanishes on the scheme-theoretic image of  $\phi^{-1}([Z]) \rightarrow X$ .

$$\begin{array}{ccc} & \tilde{D}_2(V(N), a, p(t)) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ D(V(N), a, p(t)) & & \mathcal{H}^{[2]} \end{array}$$

We want to bound the dimension of a fiber  $\pi_2^{-1}([Z])$  for  $[Z] \in \mathcal{H}^{[2]}$ . Let  $W \subset X$  be the scheme theoretic image of  $\phi^{-1}([Z])$  in  $X$ . We claim  $\deg(W)$  is twice the degree of the schemes parameterized by  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t))$ . If  $[Z]$  corresponds to two reduced points in  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t))$ , then this is clear. If  $[Z]$  corresponds to a tangent vector, we apply Lemma B.2.

Notice that we are using the fact that we have restricted  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t))$  to integral subschemes, otherwise we could for example have a deformation that just moves an embedded point around. Similarly, in the case  $[Z]$  corresponds to two reduced points, we could have the varieties above those two points coincide except for embedded points.

Now, we apply Corollary A.3, to see the dimension  $\pi_2^{-1}([Z])$  is bounded above by  $p_V(N) - Q(N)$  for a polynomial  $Q(t)$  with the same degree as  $p(t)$  with and twice the leading coefficient. Therefore,

$$\dim(\pi_1(\tilde{D}_2(V(N), a, p(t)))) \leq \dim(\mathcal{H}^{[2]}) + p_V(N) - Q(N)$$

$$\dim(\pi^{-1}(\pi_1(\tilde{D}_2(V(N), a, p(t)))) \leq \dim(\mathcal{H}^{[2]}) + p_V(N) - Q(N) + \dim(\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t)))$$

for all  $N > 0$ . Applying Proposition 4.1 shows that this is less than  $\dim(\tilde{D}(V(N), a, p(t)))$  for  $N$  large, so

$$\pi : \tilde{D}(V(N), a, p(t)) \rightarrow D(V(N), a, p(t))$$

is generically finite. Finally, we apply Proposition 4.1 again to see

$$\dim(D(V(N), a, p(t))) - \dim(\pi_1(\tilde{D}_2(V(N), a, p(t)))) \geq \dim(\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t))) - \dim(\mathcal{H}^{[2]}) - p(N) + Q(N),$$

and the right side is a polynomial with same degree and leading coefficient as  $p(t)$ , as desired. To finish, we let  $E(N)$  be the closure of  $\pi_1(\tilde{D}_2(V(N), a, p(t)))$ .  $\square$

Similarly, we also want to bound away the dimension of the intersection  $D(V(N), a, p(t)) \cap D(V(N), a, q(t))$  for two polynomials  $p(t) \neq q(t)$ .

**Proposition 5.2.** *Given  $p(t) \neq q(t)$  of degree  $\dim(X) - \text{rank}(V) + a$ , there exists  $N_0$  depending on  $p(t)$  such that for all  $N \geq N_0$ , the codimension of*

$$D(V(N), a, p(t)) \cap D(V(N), a, q(t)) \subset H^0(V(N))$$

*is bounded below by  $P(N)$  for some polynomial  $P$  of the same degree of  $p(t)$  and  $q(t)$  whose leading coefficient is the sum of the leading coefficients of  $p(t)$  and  $q(t)$ .*

Proposition 5.2 is proven in a similar way to Proposition 5.1 and without the complication of having to consider tangent vectors in the fibers, so we omit the proof. In this case, we would want to consider the incidence correspondence of triples

$$(s, [Z_1], [Z_2]) \in H^0(V(N)) \times \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t)) \times \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, q(t))$$

where  $s$  vanishes on both  $Z_1$  and  $Z_2$ . Then, like in the proof of Proposition 5.1, we use the fact that  $\deg(Z_1 \cup Z_2) = \deg(Z_1) + \deg(Z_2)$  and apply Corollary A.3. Equivalently, we could have repeated the argument of Proposition 5.1 on

$$\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t)) \cup \widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, q(t))$$

and proved both Proposition 5.1 and Proposition 5.2 at the same time.

## 6. CASE OF HIGH DEGREE

We now deal with the crux of the argument. First, since vector bundles in general don't have a filtration by bundles of smaller rank, we will be naturally led to deal with coherent sheaves that are vector bundles away from a set of high codimension.



**Definition 6.1.** If  $F$  is a coherent sheaf on  $X$  that is a vector bundle of constant rank on  $X \setminus Z$  for some closed subset  $Z \subset X$  of codimension at least  $\text{rank}(F|_{X \setminus Z}) - a + 1$ , then  $D(F, a) \subset H^0(F)$  is defined to be the locus

$$\{s \in H^0(F) \mid \dim(\{s|_{X \setminus Z} = 0\}) \geq \dim(X) - \text{rank}(F|_{X \setminus Z}) + a\}.$$

Lemma B.1 shows  $D(F, a)$  is closed.

**Remark 1.** Regarding notation, if  $X$  is integral, then instead of saying  $F$  is a vector bundle away from a closed subset of small dimension, we could instead define

$$S(F) := \{x \in X \mid \text{rank}(F|_x) > \text{rank}(F)\}$$

to be the closed subset where  $F$  jumps rank and say  $S(F)$  has small dimension. Here,  $S(F)$  is the locus where  $F$  is not locally free and is called the *singularity set* [12, Chapter 2, Section 1]. However, since our argument doesn't depend on whether  $X$  is reduced or irreducible, we will instead always refer to a large open set on which  $F$  is a vector bundle, instead of just taking the complement of  $S(F)$ .

### 6.1. A short exact sequence.

**Proposition 6.1.** *If  $F$  is a globally generated sheaf on  $X$  and is a vector bundle of constant rank on  $X \setminus Z$  for some closed subset  $Z \subset X$  of codimension at least  $\text{rank}(F|_{X \setminus Z})$ , then there exists a short exact sequence*

$$0 \rightarrow \mathcal{O}_X \rightarrow F \rightarrow Q \rightarrow 0,$$

where  $Q$  is a vector bundle of rank  $\text{rank}(F|_{X \setminus Z}) - 1$  away from some closed subset of codimension at least  $\text{rank}(F|_{X \setminus Z})$ .

*Proof.* The ideas are all in [6, Example 12.1.11], but we will describe the minor modifications necessary. Since  $F$  is globally generated, we can find a surjection  $\mathcal{O}_X^M \rightarrow F$ . If we restrict to  $X \setminus Z$ , then we can regard this surjection as a surjection of affine bundles  $\rho : (X \setminus Z) \times \mathbb{A}^M \rightarrow |(F|_{X \setminus Z})|$ . Let  $W \subset |(F|_{X \setminus Z})|$  be the zero section. Following the argument in [6, Example 12.1.11], we can find  $(t_1, \dots, t_M) \in K^M$  such that  $(T_1 - t_1, \dots, T_M - t_M)$  form a regular sequence on the pullback  $\rho^{-1}(W)$ . If the surjection  $\mathcal{O}_X^M \rightarrow F$  is given by sections  $s_1, \dots, s_M \in H^0(F)$ , then we choose our map  $\mathcal{O}_X \rightarrow F$  to be the section  $s = t_1 s_1 + \dots + t_M s_M$ .

Therefore, the vanishing locus of our section  $s \in H^0(F)$  in  $X \setminus Z$  is codimension at least  $\text{rank}(F)$ . Without loss of generality, we can enlarge  $Z$  so that  $s$  does not vanish on  $Z$ . Then, the quotient  $Q$  is a vector bundle on  $X \setminus Z$ . This is because locally on open  $U \subset X \setminus Z$  small enough, the map  $\mathcal{O}_{X \setminus Z} \xrightarrow{s} F_{X \setminus Z} \cong \mathcal{O}_{X \setminus Z}^{\text{rank}(F)}$  is described by  $1 \rightarrow (v_1, \dots, v_{\text{rank}(F)})$ . At each point of  $U$ ,  $v_i$  is not zero for some  $i$ . We can shrink  $U$  and assume  $v_1$  is nowhere zero on  $U$ . Then,  $v_1$  is invertible, and we can change basis, so the map  $\mathcal{O}_{X \setminus Z} \xrightarrow{s} F_{X \setminus Z} \cong \mathcal{O}_{X \setminus Z}^{\text{rank}(F)}$  is given by  $1 \rightarrow (1, 0, \dots, 0)$ , so  $Q$  is a vector bundle on  $X \setminus Z$ .  $\square$

**6.2. Projecting onto the quotient.** The purpose of Section 6 is to prove the following two propositions, whose proofs are almost identical. We could have also defined the appropriate generalization to Definitions 3.5 and 3.6 analogous to how Definition 6.1 generalizes Definition 3.4 and proved both results in one statement, but this seemed to make the exposition less clear.

**Proposition 6.2.** *If  $F$  is a globally generated sheaf on  $X$  and is a vector bundle of constant rank on  $X \setminus Z$  for some closed subset  $Z \subset X$  of codimension at least  $\text{rank}(F|_{X \setminus Z}) - a + 1$ , then there exist constants  $N_0, N_1$  such that the codimension of  $D(F(N), a) \subset H^0(F(N))$  is at least*

$$\frac{1}{(\dim(X) - \text{rank}(F|_{X \setminus Z}) + a)!} (N - N_1)^{\dim(X) - \text{rank}(F|_{X \setminus Z}) + a}$$

for all  $N \geq N_0$ .

*Proof.* For the base case, if  $\text{rank}(F|_{X \setminus Z}) = 0$ , then  $D(F(N), a) = \emptyset$  by definition, so the codimension is infinity according to our convention. This is the key place where we use  $a > 0$ . Now, we will use induction on  $\text{rank}(F|_{X \setminus Z})$ .

Apply Proposition 6.1 to get a short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow F \rightarrow Q \rightarrow 0,$$

where  $F$  and  $Q$  are vector bundles on  $X \setminus Z'$ , where  $Z' \supset Z$  is codimension at least  $\text{rank}(F|_{X \setminus Z}) - a + 1$  in  $X$ . Replace  $Z$  by  $Z'$ .

Pick  $N'_1$  to be large enough so that  $\mathcal{O}_X(N'_1)$  is 0-regular, so in particular the projection  $\pi : H^0(F(N)) \rightarrow H^0(Q(N))$  is surjective for  $N \geq N'_1$ . We set  $N_0 = N_1 = N'_1$  for now, and will possibly increase  $N_0$  and  $N_1$  in the remainder of the proof. We write

$$D(F(N), a) = \pi^{-1}(D(Q(N), a)) \cup (D(F(N), a) \setminus \pi^{-1}(D(Q(N), a))).$$

The codimension of  $\pi^{-1}(D(Q(N), a))$  in  $H^0(F(N))$  is the codimension of  $D(Q(N), a)$  in  $H^0(Q(N))$ , which is at least

$$\frac{1}{(\dim(X) - \text{rank}(F|_{X \setminus Z}) + a + 1)!} (N - N_1)^{\dim(X) - \text{rank}(F|_{X \setminus Z}) + a + 1}$$

for  $N \geq N_0$  (possibly after increasing  $N_0, N_1$ ) by induction on the rank of  $F$ .

Therefore, it suffices to bound the codimension of  $(D(F(N), a) \setminus \pi^{-1}(D(Q(N), a)))$ . We will do this by bounding each fiber of

$$\pi|_{D(F(N), a) \setminus \pi^{-1}(D(Q(N), a))} : D(F(N), a) \setminus \pi^{-1}(D(Q(N), a)) \rightarrow H^0(Q(N)).$$

Suppose  $s \in D(F(N), a) \setminus \pi^{-1}(D(Q(N), a))$ . Let  $W \subset X$  be the set-theoretic closure of  $\{\pi(s)|_{X \setminus Z} = 0\} \subset X \setminus Z$  in  $X$ . By assumption,  $\dim(W) = \dim(X) - \text{rank}(F) + a$ . Let  $W_1, \dots, W_\ell$  be the components of  $W$  of maximal dimension equipped with the reduced subscheme structure. Consider the short exact sequence

$$0 \rightarrow \mathcal{O}_X(N) \rightarrow F(N) \rightarrow Q(N) \rightarrow 0.$$

If  $s_1, s_2 \in \pi^{-1}(\pi(s))$ , then  $s_1 - s_2$  is in the subspace  $H^0(\mathcal{O}_X(N)) \subset H^0(F(N))$ . If  $s_1, s_2$  both vanish on  $W_i$ , then so does  $s_1 - s_2$ . By Corollary A.3 applied to the case where the vector bundle is  $\mathcal{O}_X(N'_1)$  and the degree is 1, we see that the locus  $\{s' \in \pi^{-1}(\pi(s)) | s'|_{W_i} = 0\} \subset \pi^{-1}(\pi(s))$  is codimension at least

$$\binom{N - N'_1 - 1 + (\dim(X) - \text{rank}(F|_{X \setminus Z}) + a)}{\dim(X) - \text{rank}(F|_{X \setminus Z}) + a},$$

which is at least

$$\frac{1}{(\dim(X) - \text{rank}(F|_{X \setminus Z}) + a)!} (N - N'_1)^{\dim(X) - \text{rank}(F|_{X \setminus Z}) + a}.$$

Note, that we are bounding the Hilbert function  $h_{Z_i, \mathcal{O}_X(N'_1)}(N - N'_1)$ , not the Hilbert function  $h_{Z, \mathcal{O}_X}(N)$ .

Repeating this for each  $i$  shows

$$\pi^{-1}(\pi(s)) \cap (D(F(N), a) \setminus \pi^{-1}(D(Q(N), a))) \subset \pi^{-1}(\pi(s))$$

is codimension at least  $\frac{1}{(\dim(X) - \text{rank}(F|_{X \setminus Z}) + a)!} (N - N'_1)^{\dim(X) - \text{rank}(F|_{X \setminus Z}) + a}$ . Applying Lemma 3.1 allows us conclude that the codimension of

$$D(F(N), a) \setminus \pi^{-1}(D(Q(N), a)) \subset H^0(F(N))$$

is at least  $\frac{1}{(\dim(X) - \text{rank}(F|_{X \setminus Z}) + a)!} (N - N'_1)^{\dim(X) - \text{rank}(F|_{X \setminus Z}) + a}$ . By construction,  $N_1$  is already at least  $N'_1$ . Finally, we choose  $N_0$  large enough so that

$$\frac{1}{(\dim(X) - \text{rank}(F|_{X \setminus Z}) + a + 1)!} (N - N_1)^{\dim(X) - \text{rank}(F|_{X \setminus Z}) + a + 1}$$

is at least

$$\frac{1}{(\dim(X) - \text{rank}(F|_{X \setminus Z}) + a)!} (N - N_1)^{\dim(X) - \text{rank}(F|_{X \setminus Z}) + a}$$

for  $N \geq N_0$ . □

Recall that  $V$  is assumed to be 0-regular, so it is globally generated [11, Theorem 1.8.3]. Also, note that the  $N_0, N_1$  in the statement of Proposition 6.3 depend on  $d$ , but do *not* depend on  $p(t)$ .

**Proposition 6.3.** *Fix a degree  $d$ . There exists some  $N_0, N_1$  such that for all  $p(t)$  where  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, a, p(t))$  parameterizes subschemes of degree at least  $d$ , then for  $N \geq N_0$ , the codimension of  $D(V(N), a, p(t))$  in  $H^0(V(N))$  is at least*

$$\frac{d}{(\dim(X) - \text{rank}(V) + a)!} (N - N_1)^{\dim(X) - \text{rank}(V) + a}.$$

*Proof.* We repeat the proof of Proposition 6.2 with  $F$  replaced by  $V$ . Then, when we take the components  $W_1, \dots, W_\ell$  of  $W$  of maximal dimension, we can throw out all the components of degree less than  $d$ . Therefore, when we apply Corollary A.3, we get the codimension of  $D(V(N), a, p(t)) \setminus \pi^{-1}(D(Q(N), a)) \subset H^0(V(N))$  is at least  $d \binom{N - N'_1 - d + (\dim(X) - \text{rank}(V) + a)}{\dim(X) - \text{rank}(V) + a}$ . Increasing  $N_0, N_1$  if necessary, we can assume  $N_0, N_1 \geq N'_1 + d - 1$ , so that this is at least

$$\frac{d}{(\dim(X) - \text{rank}(V) + a)!} (N - N_1)^{\dim(X) - \text{rank}(V) + a}.$$

for  $N \geq N_0$ .

As before, increasing  $N_0, N_1$  if necessary, Proposition 6.2 applied to  $Q$  shows the codimension of  $D(Q(N), a)$  in  $H^0(Q(N))$  is at least

$$\frac{1}{(\dim(X) - \text{rank}(V) + a + 1)!} (N - N_1)^{\dim(X) - \text{rank}(V) + a + 1},$$

for all  $N \geq N_0$ .

Therefore, increasing  $N_0$  further implies the codimension of  $D(V(N), a, p(t)) \subset H^0(V(N))$  is at least  $\frac{d}{(\dim(X) - \text{rank}(V) + a)!} (N - N_1)^{\dim(X) - \text{rank}(V) + a}$  for  $N \geq N_0$ . □

## 7. CONCLUSION OF THE ARGUMENT

We now put together the pieces to finish.

**Definition 7.1.** Given two polynomials  $p(t), q(t) \in \mathbb{Q}[t]$ , we say  $p$  *dominates*  $q$  if  $\lim_{t \rightarrow \infty} p(t) - q(t) = \infty$  and  $p$  is *equivalent* to  $q$  if neither  $p$  nor  $q$  dominates the other. If  $p$  and  $q$  are equivalent, we will also write this as  $p \sim q$ .

From Chow's finiteness theorem [10, Exercise I.3.28 and Theorem I.6.3], there are only finitely many components of  $\widetilde{\text{Hilb}}_X$  parameterizing subschemes of a fixed dimension. This means there exists a sequence  $p_1, p_2, \dots \in \mathbb{Q}[t]$  such that  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p_i(t))$  is nonempty for each  $i$ ,  $p_{i+1}$  dominates  $p_i$  for each  $i$ , and for every  $p$  for which  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p(t))$  is nonempty,  $p(t) \sim p_i(t)$  for some  $i$ .

**Definition 7.2.** Let  $p_1(t), p_2(t), \dots$  be a fixed choice of polynomials in  $\mathbb{Q}[t]$  with the properties above for the rest of this section.

Theorem 1.2 and Theorem 1.4 will be implied by Theorem 7.1.

**Theorem 7.1.** For each  $m \geq 0$  and  $a > 0$ , there exists  $N_0$  such that for  $N \geq N_0$ , a component of

$$D(V(N, a)) \setminus \bigcup_{i=1}^m \bigcup_{p(t) \sim p_i(t)} D(V(N), a, p(t))$$

of maximal dimension is in  $\overline{D(V(N), a, p(t))}$  for  $p(t) \sim p_{m+1}(t)$ .

Corollary 1.5 will follow from Theorem 7.2.

**Theorem 7.2.** In  $\widehat{\mathcal{M}}_{\mathbf{L}}$ , for  $m \geq 0$ ,

$$\lim_{N \rightarrow \infty} \frac{[D(V(N), a) \setminus \bigcup_{i=1}^m \bigcup_{p(t) \sim p_i(t)} D(V(N), a, p(t))]}{\mathbf{L}^{p_V(N) - p_{m+1}(N)}} = \sum_{p(t) \sim p_{m+1}(t)} \frac{[\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}]}{\mathbf{L}^{p(N) - p_{m+1}(N)}}.$$

We will prove Theorems 7.1 and 7.2 together.

*Proof.* Let  $d_{\min}$  be the degree of the varieties parameterized by  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p_{m+1}(t))$ , so  $d_{\min}$  is the product of  $(\dim(X) - \text{rank}(V) + a)!$  and the leading coefficient of  $p_{m+1}$ . Let  $d_{\text{med}} = d_{\min} \text{rank}(V) + 1$ .

Let  $M$  be the integer where  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p_M(t))$  parameterizes varieties of degree at least  $d_{\text{med}}$ , but  $\widetilde{\text{Hilb}}_X^{\dim(X) - \text{rank}(V) + a}(V, p_{M-1}(t))$  does not. Choose  $N_0$  so that

- (1) the conclusion of Proposition 4.1 hold for all  $p(t)$  where  $p(t) \sim p_i(t)$  for  $i < M$
- (2) the conclusion of Proposition 5.1 holds for all  $p(t) \sim p_{m+1}(t)$
- (3) the conclusion of Proposition 5.2 holds for all choices of  $p(t), q(t)$ , where  $p(t) \sim p_i(t)$  for  $i \leq m$  and  $q(t) \sim p_{m+1}(t)$
- (4) the conclusion of Proposition 6.3 holds in the case  $d = d_{\text{med}}$ .

For the rest of the argument, fix  $p(t) \sim p_{m+1}(t)$ . From Proposition 5.1, we can increase  $N_0$  so that  $\tilde{D}(V(N), a, p(t)) \rightarrow D(V(N), a, p(t))$  is birational onto its image for  $N \geq N_0$ , and from Proposition 5.2, we can increase  $N_0$  so that

$$\dim(D(V, a, p(t)) \setminus \bigcup_{i=1}^m \bigcup_{p'(t) \sim p_i(t)} D(V(N), a, p'(t))) = \dim(D(V, a, p(t)))$$

for  $N \geq N_0$ . Then, for  $N \geq N_0$ ,

$$\dim(D(V, a, p(t)) \setminus \bigcup_{i=1}^m \bigcup_{p'(t) \sim p_i(t)} D(V(N), a, p'(t))) - \dim(D(V, a, q(t)))$$

is bounded below by a nonconstant polynomial with positive leading coefficient for  $N \geq N_0$  if  $q(t) \sim p_i(t)$  for  $m+1 < i < M$ . Similarly, Proposition 6.3 shows that the same statement is true for  $q(t) \sim p_i(t)$  for  $i \geq M$ . This shows Theorem 7.1.

To see Theorem 7.2, we note in addition by Proposition 5.2 that

$$\dim(D(V, a, p(t)) - \dim(D(V, a, p(t)) \cap D(V(N), a, q(t)))$$

is bounded below by a nonconstant polynomial with positive leading coefficient for  $N \geq N_0$  if  $p(t) \neq q(t) \sim p_i(t)$  for  $i \leq m+1$ , and by Proposition 5.1 there is a set  $E(N) \subset \tilde{D}(V, a, p(t))$  such that the fibers of the map

$$\tilde{D}(V, a, p(t)) \setminus E(N) \rightarrow D(V, a, p(t))$$

are either empty or a single reduced point, and the codimension of  $E(N)$  in  $\tilde{D}(V, a, p(t))$  is bounded below by a nonconstant polynomial with positive leading coefficient. To conclude we apply Proposition C.1.  $\square$

## APPENDIX A. FACTS ABOUT HILBERT FUNCTIONS

Throughout the appendix, we assume that we are working in a projective scheme  $X$  and  $V$  is a vector bundle on  $X$  of constant rank unless otherwise specified.

**A.1. Lower bound on Hilbert function given degrees.** To give crude bounds on Hilbert function, we will generalize a well-known lemma (see [7, Lemma 3.1]).

**Lemma A.1.** *If  $H^1(V(n-1)) = 0$ ,  $Z \subset X$  is a closed subscheme and  $H$  is a hyperplane section of  $X$  that is a nonzero divisor when restricted to  $Z$ , then*

$$h_{Z,V}(n) \geq h_{Z,V}(n-1) + h_{Z \cap H, V}(n).$$

*Proof.* Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(V(n-1)) & \xrightarrow{\times H} & H^0(V(n)) & \longrightarrow & H^0(V(n)|_H) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(V(n-1)|_Z) & \xrightarrow{\times H} & H^0(V(n)|_Z) & \longrightarrow & H^0(V(n)|_{H \cap Z}) \end{array}$$

Then,

$$\begin{aligned}
h_{Z,V}(n) &:= \dim(\operatorname{im}(H^0(V(n)) \rightarrow H^0(V(n)|_Z))) \\
&= \dim(\operatorname{im}(H^0(V(n)) \rightarrow H^0(V(n)|_Z)) \cap \operatorname{im}(H^0(V(n-1)|_Z) \rightarrow H^0(V(n)_Z))) + \\
&\quad \dim(\operatorname{im}(H^0(V(n)) \rightarrow H^0(V(n)|_{H \cap Z}))) \\
&\geq \dim(\operatorname{im}(H^0(V(n-1)) \rightarrow H^0(V(n-1)|_Z))) + \dim(\operatorname{im}(H^0(V(n)) \rightarrow H^0(V(n)|_{H \cap Z}))) \\
&= h_{Z,V}(n-1) + h_{Z \cap H, V}(n).
\end{aligned}$$

□

**Lemma A.2.** *If  $V$  is 0-regular in the sense of Castelnuovo-Mumford regularity,  $Z \subset X$  a subscheme of degree at least  $d$  and dimension  $m$ , then for  $n \geq 1$*

$$h_{Z,V}(n-1) \geq \begin{cases} \operatorname{rank}(V) \binom{n+m}{m+1} & \text{if } n \leq d \\ \operatorname{rank}(V) \sum_{i=0}^m \binom{n-d+i-1}{i} \binom{d+m-i}{m-i+1} & \text{if } n > d. \end{cases}$$

*Proof.* We will use induction and Lemma A.1. First, we show the base case  $m = 0$ . Let  $Z = \operatorname{Spec}(A)$  for an Artinian  $K$ -algebra  $A$  and fix an trivialization  $V|_Z \cong \mathcal{O}_{\operatorname{Spec}(A)}^{\operatorname{rank}(V)}$ . We want to show  $h_{Z,V}(n-1) \geq \operatorname{rank}(V) \min\{n, d\}$ . Without loss of generality, we can assume  $d \leq n$ , or else we can just restrict  $Z$  to a subscheme of length  $n$ . (Recall  $A$  has quotients of any length less than  $A$ . To show this, it suffices to see that if  $A$  is local, then there is an ideal of  $A$  of length 1. To exhibit such an ideal, we can pick any element  $a \in A$  that is annihilated by the maximal ideal of  $A$  and take the ideal generated by  $a$ .) Consider

$$\begin{array}{ccccc}
H^0(V(n-1)) & \longrightarrow & H^0(V(n-1)|_Z) & \xrightarrow{\sim} & A^{\operatorname{rank}(V)} \\
\uparrow & & \uparrow & & \uparrow \\
H^0(V) \otimes H^0(\mathcal{O}_X(n-1)) & \longrightarrow & H^0(V|_Z) \otimes H^0(\mathcal{O}_Z(n-1)) & \xrightarrow{\sim} & A^{\operatorname{rank}(V)} \otimes A
\end{array}$$

Since  $V$  is globally generated [11, Theorem 1.8.3], there is a section  $s \in H^0(V)$  such that  $s|_Z \in H^0(V|_Z) \cong A^{\operatorname{rank}(V)}$  maps to  $(s_1, \dots, s_{\operatorname{rank}(V)}) \in A^{\operatorname{rank}(V)}$  such that each  $s_i \in A$  is invertible. From [9, Theorem 1.1], we know  $H^0(\mathcal{O}_X(n-1)) \rightarrow H^0(\mathcal{O}_Z(n-1))$  is surjective.

Pick any section in  $s' \in H^0(V(n-1)|_Z)$ . This maps to some  $(s'_1, \dots, s'_{\operatorname{rank}(V)})' \in A^{\operatorname{rank}(V)}$ . Pick  $a_1, \dots, a_{\operatorname{rank}(V)} \in A$  such that  $a_i = s'_i s_i^{-1}$  for each  $i$ . Then, the section

$$s_1 \otimes a_1 + \dots + s_{\operatorname{rank}(V)} \otimes a_{\operatorname{rank}(V)} \in A^{\operatorname{rank}(V)} \otimes A \cong H^0(V|_Z) \otimes H^0(\mathcal{O}_Z(n-1))$$

maps so  $s' \in H^0(V(n-1)|_Z)$ .

Next, for the induction step, suppose we know Lemma A.2 in dimension  $m-1$ . From the long exact sequence, we have the exactness of

$$H^i(V(-i)) \rightarrow H^i(V(-i)|_H) \rightarrow H^{i+1}(V(-i-1))$$

so  $V|_H$  is still 0-regular. Also,  $H^0(V) \rightarrow H^0(V|_H)$  is surjective as  $H^1(V(-1)) = 0$ . Therefore, we can apply the induction hypothesis and Lemma A.1.

If  $Z$  is dimension  $m$  and degree  $d$ , we have for  $n \leq d$

$$\begin{aligned} h_{Z,V}(n-1) &\geq h_{Z,V}(-1) + h_{Z \cap H,V}(0) + h_{Z \cap H,V}(1) + \cdots + h_{Z \cap H,V}(n-1) \\ &\geq 0 + \text{rank}(V) \binom{m}{m} + \cdots + \text{rank}(V) \binom{m+n-1}{m} = \text{rank}(V) \binom{n+m}{m+1}. \end{aligned}$$

If  $n > d$ ,

$$\begin{aligned} h_{Z,V}(n-1) &\geq h_{Z,V}(d-1) + (h_{Z \cap H,V}(d) + \cdots + h_{Z \cap V}(n-1)) \\ &= \text{rank}(V) \binom{d+m}{m+1} + \sum_{j=d+1}^n \text{rank}(V) \sum_{i=0}^{m-1} \binom{j-d+i-1}{i} \binom{d+m-1-i}{m-1-i+1} \\ &= \text{rank}(V) \binom{d+m}{m+1} + \text{rank}(V) \sum_{i=0}^{m-1} \binom{d+m-1-i}{m-1-i+1} \sum_{j=d+1}^n \binom{j-d+i-1}{i} \\ &= \text{rank}(V) \binom{d+m}{m+1} + \text{rank}(V) \sum_{i=0}^{m-1} \binom{d+m-1-i}{m-1-i+1} \binom{n-d+i}{i+1} \\ &= \text{rank}(V) \binom{d+m}{m+1} + \text{rank}(V) \sum_{i=1}^m \binom{d+m-i}{m-i+1} \binom{n-d+i-1}{i}. \end{aligned}$$

□

**Corollary A.3.** *If  $Z \subset X$  is a subscheme of degree at least  $d$  and dimension  $m$ , then for  $n \geq 0$ ,*

$$h_{Z,V}(n) \geq d \cdot \text{rank}(V) \binom{n-d+m}{m}$$

for a 0-regular vector bundle  $V$  on  $X$ .

*Proof.* We bound  $h_{Z,V}(n)$  from below by the  $i = m$  term in summation in the statement of Lemma A.2. □

**A.2. Hilbert polynomial of a coherent sheaf.** From considering hyperplane slices and applying additivity of Euler characteristic in short exact sequence, we have the following standard fact:

**Lemma A.4.** *Suppose  $Z$  is an integral projective scheme and  $F$  is a coherent sheaf on  $Z$ . Then, the Hilbert polynomial  $\chi(E(n))$  is a polynomial of degree  $\dim(Z)$  with leading coefficient  $\frac{\deg(Z)\text{rank}(F)}{\dim(Z)!}$ .*

## APPENDIX B. SCHEME THEORETIC FACTS

**Lemma B.1.** *Suppose  $X$  is an integral projective scheme and  $Z \subset X$  a closed subset. Let  $S$  be a finite type  $K$ -scheme,  $Y \subset S \times (X \setminus Z)$  be a closed subset and  $\pi : Y \rightarrow S$  be the projection. Then,*

$$\{s \in S : \dim(\pi^{-1}(s)) \geq d\}$$

is a closed subset of  $S$  for all  $d > \dim(Z)$ .

*Proof.* Let  $\bar{Y}$  be the closure of  $Y$  in  $S \times X$  and  $\pi : \bar{Y} \rightarrow S$  be the projection. Let  $Y_d \subset Y$  be the closed subset  $\{y \in Y : \dim(\pi^{-1}(\pi(y)) \cap Y) \geq d\}$ . Let  $\bar{Y}_d \subset \bar{Y}$  be the closed subset  $\{y \in \bar{Y} : \dim(\pi^{-1}(\pi(y))) \geq d\}$ .

Since  $Y_d \subset \bar{Y}_d$ ,  $\pi(Y_d) \subset \pi(\bar{Y}_d)$ . We want to show  $\pi(Y_d) = \pi(\bar{Y}_d)$ .

To see the other inclusion, we first claim  $Y_d = \bar{Y}_d \setminus (S \times Z)$ . Given this,  $\pi(Y_d) \supset \pi(\bar{Y}_d)$  also holds, as if  $s \in \pi(\bar{Y}_d)$  is a closed point, then  $\pi^{-1}(s)$  has dimension at least  $d$  in  $X$ , so  $\pi^{-1}(s) \setminus Z$  still has dimension at least  $d$  as  $\dim(Z) < d$ .

To see  $Y_d = \bar{Y}_d \setminus (S \times Z)$ , suppose  $\xi \in \bar{Y}$  is a scheme theoretic point with  $\dim(\overline{\{\xi\}}) \geq d$  that maps to a closed point of  $S$  and  $y \in \overline{\{\xi\}} \setminus Z$ . Equivalently,  $y \in \bar{Y}_d \setminus (S \times Z)$ . But then  $\overline{\{\xi\}}$  cannot be contained in  $Z$  since  $\dim(Z) < \dim(\overline{\{\xi\}})$ , so  $\xi \in Y$  and  $y \in Y_d$ .  $\square$

**Lemma B.2.** *Suppose  $Z \subset \mathbb{P}^r$  is a integral projective scheme, and  $\mathcal{Z} \subset \mathbb{P}_{\text{Spec}(K[\epsilon]/(\epsilon^2))}^r$  is a nontrivial infinitesimal deformation of the embedding  $Z \subset \mathbb{P}^r$ . Let  $Z' \subset \mathbb{P}^r$  be the scheme theoretic image of  $\mathcal{Z} \rightarrow \mathbb{P}^r$ . Then,  $\deg(Z') = 2 \deg(Z)$ .*

*Proof.* Assume  $Z$  is not contained in hyperplane  $\{X_0 = 0\}$ . In the chart  $\{X_0\}$ , the data of the embedding  $Z \subset \mathbb{P}^r$  is the surjection  $K[x_1, \dots, x_r] \rightarrow A$  for  $\text{Spec}(A) = Z \setminus \{X_0 = 0\}$  and similarly for  $\mathcal{Z} \subset \mathbb{P}_{\text{Spec}(K[\epsilon]/(\epsilon^2))}^r \rightarrow \mathcal{A}$  for  $\text{Spec}(\mathcal{A}) = \mathcal{Z} \setminus \{X_0 = 0\}$ .

$$\begin{array}{ccc} K[x_1, \dots, x_r] & \xrightleftharpoons{\quad} & K[\epsilon, x_1, \dots, x_r]/(\epsilon^2) \\ \downarrow & & \downarrow \\ A & \longleftarrow & \mathcal{A} \end{array}$$

The scheme theoretic image of  $\mathcal{Z} \rightarrow \mathbb{P}^r$  restricted to  $\{X_0 = 0\}$  can be computed affine locally [13, Tag 01R8] and is  $\text{Spec}(B)$  for  $B = \text{im}(K[x_1, \dots, x_r] \rightarrow \mathcal{A})$ . Equivalently, it is defined by the ideal  $I \subset K[x_1, \dots, x_r]$ , where  $I = \ker(K[x_1, \dots, x_r] \rightarrow \mathcal{A})$ . Let  $\eta \in \text{Spec}(A)$  be the generic point. It suffices to show the multiplicity of  $B$  at  $\eta$  is 2, where multiplicity is defined to be the length of  $B_\eta$  [6, Section 1.5]. In the argument below, we will use  $A$  is integral to see  $A \rightarrow A_\eta$  is injective.

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & K[x_1, \dots, x_r] & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_\eta & \longrightarrow & K[x_1, \dots, x_r]_\eta & \longrightarrow & B_\eta \longrightarrow 0 \end{array}$$

From exactness of localization, it suffices to show that the induced map  $B_\eta \rightarrow A_\eta$  is not an isomorphism. We will assume  $B_\eta \rightarrow A_\eta$  is an isomorphism, and we will show that the deformation  $\text{Spec}(\mathcal{A}) \rightarrow \text{Spec}(K[\epsilon]/(\epsilon^2))$  must be trivial.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \epsilon A_\eta & \longrightarrow & \mathcal{A}_\eta & \longrightarrow & A_\eta \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \epsilon A & \longrightarrow & \mathcal{A} & \longrightarrow & A \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \epsilon K & \longrightarrow & K[\epsilon]/(\epsilon^2) & \longrightarrow & K \longrightarrow 0 \end{array}$$



The middle row is exact by flatness [8, Proposition 2.2]. From the diagram, we see  $\mathcal{A} \rightarrow \mathcal{A}_\eta$  is an injection.

$$\begin{array}{ccccc}
 B & \xrightarrow{\quad} & \mathcal{A} & \xrightarrow{\quad} & A & K[\epsilon, x_1, \dots, x_r]/(\epsilon^2) & \twoheadrightarrow & \mathcal{A} \\
 \downarrow & & \downarrow & & \downarrow & \updownarrow & & \updownarrow \\
 B_\eta & \xrightarrow{\quad} & \mathcal{A}_\eta & \xrightarrow{\quad} & A_\eta & K[x_1, \dots, x_r] & \twoheadrightarrow & A \\
 & \searrow \sim & & & & & & \swarrow \\
 & & & & & & & A \otimes_K K[\epsilon]/(\epsilon^2)
 \end{array}$$

Since  $B \rightarrow A$  is surjective, and the composition  $B \rightarrow A \rightarrow A_\eta$  is injective in the diagram above,  $B \rightarrow A$  is an isomorphism. This gives a splitting  $A \rightarrow \mathcal{A}$  compatible with the splitting  $K[x_1, \dots, x_r] \rightarrow K[\epsilon, x_1, \dots, x_r]/(\epsilon^2)$ . This shows that  $\text{Spec}(\mathcal{A}) \rightarrow \text{Spec}(K[\epsilon]/(\epsilon^2))$  is the trivial deformation.  $\square$

### APPENDIX C. ISOMORPHISM IN THE GROTHENDIECK RING

Separability is important in positive characteristic and this is guaranteed by the assumption that all of our fibers are reduced points.

**Proposition C.1.** *Suppose  $f : Y \rightarrow Z$  is a morphism of finite type  $K$ -schemes and  $A \subset Y$  and  $B \subset Z$  are constructible sets such that  $f(A) = B$  and  $f^{-1}(p)$  is a single reduced point for each closed  $p \in B$ . Then  $[A] = [B]$  in the Grothendieck ring  $\mathcal{M}$ .*

*Proof.* By assumption  $f$  induces a bijection  $A \rightarrow B$ . If we write  $B = U_1 \cup \dots \cup U_\ell$  as a finite union of locally closed sets, it suffices to show  $[f^{-1}(U_i)] = [U_i]$ . Therefore, it suffices to show the case where  $B$  is locally closed. Similarly, we can break  $B$  up further, so that  $B$  is irreducible and locally closed.

Since  $B$  is locally closed,  $A$  is also locally closed. We can equip  $A$  and  $B$  with the reduced subscheme structure of an open subscheme. Then,  $f : A \rightarrow B$  is a map of irreducible finite type  $K$ -schemes such that each fiber over a closed point  $p \in B$  is a single reduced point. Pick an affine open  $\text{Spec}(R_B) \subset B$  and an affine open  $\text{Spec}(R_A) \subset f^{-1}(\text{Spec}(R_B))$ .

By Grothendieck's generic freeness lemma [5, Theorem 14.4], we can restrict  $R_B$  so that  $R_A$  is free over  $R_B$ . By the assumption on the fibers of  $f : A \rightarrow B$ ,  $R_A$  must be rank at most 1 over  $R_B$ , so the map  $R_B \rightarrow R_A$  is an isomorphism. Then,  $\text{Spec}(R_A) \rightarrow \text{Spec}(R_B)$  is an isomorphism and we use Noetherian induction on the complement  $f : A \setminus \text{Spec}(R_A) \rightarrow B \setminus \text{Spec}(R_B)$ .  $\square$

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